14-7 Solving Trigonometric Equations

Main Ideas

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

New Vocabulary

trigonometric equations

GET READY for the Lesson

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function $T = 11.56 \sin (0.4516x - 1.641) + 80.89$, where T represents the average daily high temperature in degrees Fahrenheit and x represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

Solve Trigonometric Equations You have seen that trigonometric identities are true for *all* values of the variable for which the equation is defined. However, most **trigonometric equations**, like some algebraic equations, are true for *some* but not *all* values of the variable.

EXAMPLE 1 Solve Equations for a Given Interval

Find all solutions of $\sin 2\theta = 2 \cos \theta$ for the interval $0 \leq \theta < 360°$.

$$\sin 2\theta = 2\cos\theta \quad \text{Original equation}$$

$$2\sin\theta\cos\theta = 2\cos\theta \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$2\sin\theta\cos\theta - 2\cos\theta = 0 \quad \text{Solve for 0.}$$

$$2\cos\theta(\sin\theta - 1) = 0 \quad \text{Factor.}$$

Use the Zero Product Property.

$2\cos\theta = 0$ or $\sin\theta - 1 = 0$

$\cos\theta = 0$ $\qquad$ $\sin\theta = 1$

$\theta = 90°$ or $270°$ $\qquad$ $\theta = 90°$

The solutions are 90° and 270°.

CHECK Your Progress

1. Find all solutions of $\cos^2\theta = 1$ for the interval $0° \leq \theta < 360°$.

Trigonometric equations are usually solved for values of the variable between 0° and 360° or 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

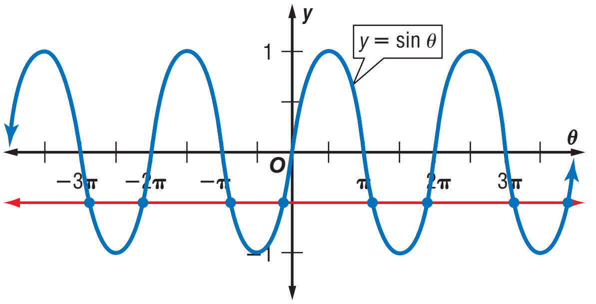
EXAMPLE Solve Trigonometric Equations

2 **Solve $2 \sin \theta = -1$ for all values of θ if θ is measured in radians.**

$2 \sin \theta = -1$ Original equation

$\sin \theta = -\frac{1}{2}$ Divide each side by 2.

Look at the graph of $y = \sin \theta$ to find solutions of $\sin \theta = -\frac{1}{2}$.



The solutions are $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$, and so on, and $\frac{-7\pi}{6}, \frac{-11\pi}{6}, \frac{-19\pi}{6}, \frac{-23\pi}{6}$, and so on. The only solutions in the interval 0 to 2π are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is 2π radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

Study Tip

Expressing Solutions as Multiples

The expression $\frac{\pi}{2} + k \cdot \pi$ includes $\frac{3\pi}{2}$ and its multiples, so it is not necessary to list them separately,

CHECK Your Progress

2. Solve for $\cos 2\theta + \cos \theta + 1 = 0$ for all values of θ if θ is measured in degrees.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

EXAMPLE Solve Trigonometric Equations Using Identities

3 **Solve $\cos \theta \tan \theta - \sin^2 \theta = 0$.**

$\cos \theta \tan \theta - \sin^2 \theta = 0$ Original equation

$\cos \theta \left(\frac{\sin \theta}{\cos \theta}\right) - \sin^2 \theta = 0$ $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\sin \theta - \sin^2 \theta = 0$ Multiply.

$\sin \theta (1 - \sin \theta) = 0$ Factor.

$\sin \theta = 0$ or $1 - \sin \theta = 0$

$\theta = 0°, 180°, \text{ or } 360°$ $\quad \sin \theta = 1$

$\theta = 90°$

CHECK

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 0° \tan 0° - \sin^2 0° \stackrel{?}{=} 0$ $\theta = 0°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 180° \tan 180° - \sin^2 180° \stackrel{?}{=} 0$ $\theta = 180°$

$-1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 360° \tan 360° - \sin^2 360° \stackrel{?}{=} 0$ $\theta = 360°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 90° \tan 90° - \sin^2 90° \stackrel{?}{=} 0$ $\theta = 90°$

$\tan 90°$ is undefined.

Thus, 90° is not a solution.

The solution is $0° + k \cdot 180°$.

CHECK Your Progress

Solve each equation.

3A. $\sin\theta\cot\theta - \cos^2\theta = 0$

3B. $\dfrac{\cos\theta}{\cot\theta} + 2\sin^2\theta = 0$

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Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between -1 and 1, inclusive. Thus, the solution set for $\cos x = 4$ is empty.

EXAMPLE 4 Determine Whether a Solution Exists

Solve $3\cos 2\theta - 5\cos\theta = 1$.

$3\cos 2\theta - 5\cos\theta = 1$ Original equation

$3(2\cos^2\theta - 1) - 5\cos\theta = 1$ $\cos 2\theta = 2\cos^2\theta - 1$

$6\cos^2\theta - 3 - 5\cos\theta = 1$ Multiply.

$6\cos^2\theta - 5\cos\theta - 4 = 0$ Subtract 1 from each side.

$(3\cos\theta - 4)(2\cos\theta + 1) = 0$ Factor.

$3\cos\theta - 4 = 0$ or $2\cos\theta + 1 = 0$

$3\cos\theta = 4$ $\quad$ $2\cos\theta = -1$

$\cos\theta = \frac{4}{3}$ $\quad$ $\cos\theta = -\frac{1}{2}$

Not possible since $\cos\theta$ cannot be greater than 1. $\quad$ $\theta = 120°$ or $240°$

Thus, the solutions are $120° + k \cdot 360°$ and $240° + k \cdot 360°$.

CHECK Your Progress **Solve each equation.**

4A. $\sin^2\theta + 2\cos^2\theta = 4$

4B. $\cos^2\theta - 3 = 4 - \sin^2\theta$

Use Trigonometric Equations Trigonometric equations are often used to solve real-world situations.

Real-World EXAMPLE

5 **GARDENING** **Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight H in her town can be represented by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?**

$H = 11.45 + 6.5 \sin (0.0168d - 1.333)$ Original equation

$14 = 11.45 + 6.5 \sin (0.0168d - 1.333)$ $H = 14$

$2.55 = 6.5 \sin (0.0168d - 1.333)$ Subtract 11.45 from each side.

$0.392 = \sin (0.0168d - 1.333)$ Divide each side by 6.5.

$0.403 = 0.0168d - 1.333$ $\sin^{-1} 0.392 = 0.403$

$1.736 = 0.0168d$ Add 1.333 to each side.

$103.333 = d$ Divide each side by 0.0168.

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

CHECK Your Progress

5. If Rhonda decides to wait only until there are 12 hours of daylight, on what day is it safe for her to plant her flowers?

CHECK Your Understanding

Example 1 (p. 861)

Find all solutions of each equation for the given interval.

1. $4 \cos^2 \theta = 1; 0° \leq \theta < 360°$

2. $2 \sin^2 \theta - 1 = 0; 90° < \theta < 270°$

3. $\sin 2\theta = \cos \theta; 0 \leq \theta < 2\pi$

4. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \leq \theta < \frac{\pi}{2}$

Example 2 (p. 862)

Solve each equation for all values of θ if θ is measured in radians.

5. $\cos 2\theta = \cos \theta$

6. $\sin \theta + \sin \theta \cos \theta = 0$

Solve each equation for all values of θ if θ is measured in degrees.

7. $\sin \theta = 1 + \cos \theta$

8. $2 \cos^2 \theta + 2 = 5 \cos \theta$

Examples 3, 4 (pp. 862–863)

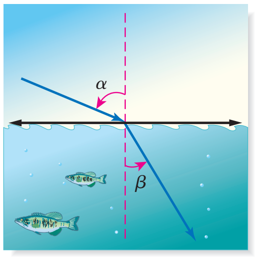
Solve each equation for all values of θ.

9. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

10. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

Example 5 (p. 864)

11. **PHYSICS** According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water?



Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–15	1
16–23	2
24–27	3, 4
28, 29	5

Find all solutions of each equation for the given interval.

12. $2\cos\theta - 1 = 0;\ 0° \le \theta < 360°$

13. $2\sin\theta = -\sqrt{3};\ 180° < \theta < 360°$

14. $4\sin^2\theta = 1;\ 180° < \theta < 360$

15. $4\cos^2\theta = 3;\ 0° \le \theta < 360°$

Solve each equation for all values of θ if θ is measured in radians.

16. $\cos 2\theta + 3\cos\theta - 1 = 0$

17. $2\sin^2\theta - \cos\theta - 1 = 0$

18. $\cos^2\theta - \frac{5}{2}\cos\theta - \frac{3}{2} = 0$

19. $\cos\theta = 3\cos\theta - 2$

Solve each equation for all values of θ if θ is measured in degrees.

20. $\sin\theta = \cos\theta$

21. $\tan\theta = \sin\theta$

22. $\sin^2\theta - 2\sin\theta - 3 = 0$

23. $4\sin^2\theta - 4\sin\theta + 1 = 0$

Solve each equation for all values of θ.

24. $\sin^2\theta + \cos 2\theta - \cos\theta = 0$

25. $2\sin^2\theta - 3\sin\theta - 2 = 0$

26. $\sin^2\theta = \cos^2\theta - 1$

27. $2\cos^2\theta + \cos\theta = 0$

WAVES **For Exercises 28 and 29, use the following information.**

After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h\sin\frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds.

28. If $h = 3$ and $P = 2$, write the equation for the wave and draw its graph over a 10-second interval.

29. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

Find all solutions of each equation for the given interval.

30. $2\cos^2\theta = \sin\theta + 1;\ 0 \le \theta < 2\pi$

31. $\sin^2\theta - 1 = \cos^2\theta;\ 0 \le \theta < \pi$

32. $2\sin^2\theta + \sin\theta = 0;\ \pi < \theta < 2\pi$

33. $2\cos^2\theta = -\cos\theta;\ 0 \le \theta < 2\pi$

Solve each equation for all values of θ if θ is measured in radians.

34. $4\cos^2\theta - 4\cos\theta + 1 = 0$

35. $\cos 2\theta = 1 - \sin\theta$

36. $(\cos\theta)(\sin 2\theta) - 2\sin\theta + 2 = 0$

37. $2\sin^2\theta + (\sqrt{2} - 1)\sin\theta = \frac{\sqrt{2}}{2}$

Solve each equation for all values of θ if θ is measured in degrees.

38. $\tan^2\theta - \sqrt{3}\tan\theta = 0$

39. $\cos^2\theta - \frac{7}{2}\cos\theta - 2 = 0$

40. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3}\sin\theta + \cos\theta$

41. $1 - \sin^2\theta - \cos\theta = \frac{3}{4}$

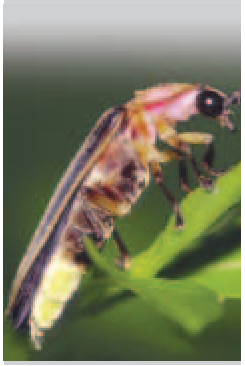
Solve each equation for all values of θ.

42. $\sin\frac{\theta}{2} + \cos\theta = 1$

43. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{2}$

44. $2\sin\theta = \sin 2\theta$

45. $\tan^2\theta + \sqrt{3} = \left(1 + \sqrt{3}\right)\tan\theta$



Real-World Link

Fireflies are bioluminescent, which means that they produce light through a biochemical reaction. Almost 100% of a firefly's energy is given off as light.

Source: www.nfs.gov

LIGHT **For Exercises 46 and 47, use the following information.**

The height of the International Peace Memorial at Put-in-Bay, Ohio, is 352 feet.

46. The length of the shadow S of the Memorial depends upon the angle of inclination of the Sun, θ. Express S as a function of θ.

47. Find the angle of inclination θ that will produce a shadow 560 feet long.

H.O.T. Problems

EXTRA PRACTICE
See pages 924, 939.

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48. OPEN ENDED Write an example of a trigonometric equation that has no solution.

49. REASONING Explain why the equation $\sec \theta = 0$ has no solutions.

50. CHALLENGE Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$. If the coordinates of an image point are (3, 4) after a 60° rotation, what are the coordinates of the preimage point?

51. REASONING Explain why the number of solutions to the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is infinite.

52. *Writing in Math* Use the information on page 861 to explain how trigonometric equations can be used to predict temperature. Include an explanation of why the sine function can be used to model the average daily temperature and an explanation of why, during one period, you might find a specific average temperature twice.

STANDARDIZED TEST PRACTICE

53. ACT/SAT Which of the following is *not* a possible solution of $0 = \sin \theta + \cos \theta \tan^2 \theta$?

A $\frac{3\pi}{4}$

B $\frac{7\pi}{4}$

C 2π

D $\frac{5\pi}{2}$

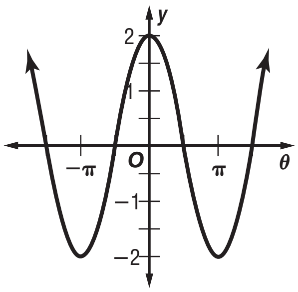
54. REVIEW The graph of the equation $y = 2 \cos \theta$ is shown. Which is a solution for $2 \cos \theta = 1$?

F $\frac{8\pi}{3}$

G $\frac{13\pi}{3}$

H $\frac{10\pi}{3}$

J $\frac{15\pi}{3}$



Spiral Review

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following. (Lesson 14-6)

55. $\sin \theta = \frac{3}{5}$; $0° < \theta < 90°$

56. $\cos \theta = \frac{1}{2}$; $0° < \theta < 90°$

57. $\cos \theta = \frac{5}{6}$; $0° < \theta < 90°$

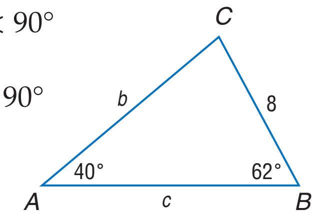
58. $\sin \theta = \frac{4}{5}$; $0° < \theta < 90°$

Find the exact value of each expression. (Lesson 14-5)

59. $\sin 240°$

60. $\cos 315°$

61. $\sin 150°$



62. Solve $\triangle ABC$. Round measures of sides and angles to the nearest tenth. (Lesson 13-4)

Cross-Curricular Project

Algebra and Physics

So you want to be a rocket scientist? It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

Math Online Cross-Curricular Project at algebra2.com